State feedback for overdetermined 2D systems: Pole placement for bundle maps over an algebraic curve.

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29 June 2009

Here is the plan of my lecture:

1. Definition of **Operator vessel**
2. Transfer functions of operator vessels
3. The discriminant curve of a vessel
4. State feedback and the pole placement problem
5. The solution
6. Conclusions

The definition of operator vessels

We start with an Overdetermined 2D continuous-time time-invariant linear i/s/o system Σ

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + B_1 u(t_1, t_2) \\ \\ \frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + B_2 u(t_1, t_2) \\ \\ y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2) \end{cases}$$

$u(t_1, t_2) \in \mathcal{E}$ - input space. $x(t_1, t_2) \in \mathcal{H}$ - state space. $y(t_1, t_2) \in \mathcal{E}_*$ - output space. All spaces are finite dimensional over the complex numbers.
$A_1, A_2 : \mathcal{H} \to \mathcal{H}$. $B_1, B_2 : \mathcal{E} \to \mathcal{H}$. $C : \mathcal{H} \to \mathcal{E}_*$. $D : \mathcal{E} \to \mathcal{E}_*$.

The definition of operator vessels - A1

Assuming x is smooth, we have $\frac{\partial}{\partial t_1}\frac{\partial x}{t_2} = \frac{\partial}{\partial t_2}\frac{\partial x}{t_1}$, so that from Σ, we have:

$$A_1\frac{\partial x}{\partial t_2} + B_1\frac{\partial u}{\partial t_2} = A_2\frac{\partial x}{\partial t_1} + B_2\frac{\partial u}{\partial t_1}$$

Replacing $\frac{\partial x}{\partial t_i}$ with the terms in Σ, we obtain

$$A_1[A_2x + B_2u] + B_1\frac{\partial u}{\partial t_2} = A_2[A_1x + B_1u] + B_2\frac{\partial u}{\partial t_1} \tag{1.1}$$

Setting $u = 0$, we see that we must have $A_1A_2 = A_2A_1$. Hence, we require our systems to satisfy this compatibility condition:

$$(A1) \qquad A_1A_2 = A_2A_1$$

The definition of operator vessels - A2

Under the assumption (A1), (1.1) becomes

$$B_2\frac{\partial u}{\partial t_1} - B_1\frac{\partial u}{\partial t_2} + (A_2B_1 - A_1B_2)u = 0 \tag{1.2}$$

We now choose an auxiliary Hilbert space $\tilde{\mathcal{E}}$ and a factorization

$$B_2 = \tilde{B}\sigma_2 \quad B_1 = \tilde{B}\sigma_1 \quad A_2B_1 - A_1B_2 = \tilde{B}\gamma \tag{1.3}$$

where

$$\tilde{B} : \tilde{\mathcal{E}} \to \mathcal{H} \quad \sigma_1 : \mathcal{E} \to \tilde{\mathcal{E}} \quad \sigma_2 : \mathcal{E} \to \tilde{\mathcal{E}} \quad \gamma : \mathcal{E} \to \tilde{\mathcal{E}}$$

In terms of this factorization, (1.3) becomes our second compatibility condition:

$$(A2) \quad A_2\tilde{B}\sigma_1 - A_1\tilde{B}\sigma_2 = \tilde{B}\gamma$$

The definition of operator vessels

Using this factorization, our equation (1.2) becomes

$$\tilde{B}[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0 \tag{1.4}$$

A sufficient condition for this to hold is

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0 \tag{1.5}$$

The definition of operator vessels

Further analysis of this system leads us to the notion of a **Livsic-Kravitsky commutative operator vessel**:
An operator vessel $\mathcal{B}$ is a collection of operators and spaces

$$\mathcal{B} = (A_1, A_2, \tilde{B}, C, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*; \mathcal{H}, \mathcal{E}, \mathcal{E}_*, \tilde{\mathcal{E}}, \tilde{\mathcal{E}}_*)$$

satisfying:

$$\begin{array}{cc}
(A1) & A_1 A_2 = A_2 A_1 \\
(A2) & A_2 \tilde{B} \sigma_1 - A_1 \tilde{B} \sigma_2 = \tilde{B} \gamma \\
(A3) & \sigma_{2*} C A_1 - \sigma_{1*} C A_2 + \gamma_* C = 0 \\
(A4) & \sigma_{1*} D = \tilde{D} \sigma_1 \quad \sigma_{2*} D = \tilde{D} \sigma_2 \\
 & \gamma_* D = \tilde{D} \gamma + \sigma_{1*} C \tilde{B} \sigma_2 - \sigma_{2*} C \tilde{B} \sigma_1
\end{array}$$

The definition of operator vessels

we also require the functions u to satisfy the following PDEs

$$[\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + \gamma] u(t_1, t_2) = 0$$

which implies that y satisfies

$$[\sigma_{2*} \frac{\partial}{\partial t_1} - \sigma_{1*} \frac{\partial}{\partial t_2} + \gamma_*] y(t_1, t_2) = 0$$

The system of equations associated to a vessel $\mathcal{B}$ is

$$\begin{cases} \frac{\partial x}{\partial t_1}(t_1, t_2) = A_1 x(t_1, t_2) + \tilde{B}\sigma_1 u(t_1, t_2) \\ \frac{\partial x}{\partial t_2}(t_1, t_2) = A_2 x(t_1, t_2) + \tilde{B}\sigma_2 u(t_1, t_2) \\ y(t_1, t_2) = C x(t_1, t_2) + D u(t_1, t_2) \end{cases}$$

Transfer functions of operator vessels

We construct the transfer function of a vessel $\mathcal{B}$ using frequency domain analysis. Suppose

$$u(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} u_0$$
$$x(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} x_0$$

and

$$y(t_1, t_2) = e^{\lambda_1 t_1 + \lambda_2 t_2} y_0$$

for some $u_0 \in \mathcal{E}$, $x_0 \in \mathcal{H}$ and $y_0 \in \mathcal{E}_*$.

Transfer functions of operator vessels

From the PDE satisfied by the input function u, we obtain an algebraic equation

$$[\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma]u_0 = 0$$

and similarly, for the output

$$[\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*]y_0 = 0$$

Plugging u, x and y into Σ, we obtain the following system of equations

$$\begin{aligned} \lambda_1 x_0 &= A_1 x_0 + \tilde{B}\sigma_1 u_0 \\ \lambda_2 x_0 &= A_2 x_0 + \tilde{B}\sigma_2 u_0 \\ y_0 &= C x_0 + D u_0 \end{aligned}$$

Transfer functions of operator vessels

To solve this system (i.e, to find y_0 in terms of u_0), we multiply the first equation by $\xi_1 \in \mathbb{C}$, the second by $\xi_2 \in \mathbb{C}$. Adding the resulting equations, we obtain

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \tilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$$

Recall that the joint spectrum of a pair of commuting square matrices $A, B \in M_n(\mathbb{C})$, denoted by $Spec(A, B)$, is the set of pairs (λ, μ) such that $\exists 0 \neq v \in \mathbb{C}^n$, with $Av = \lambda v$ and $Bv = \mu v$.

Lemma 2.1. *Given $A, B \in M_n(\mathbb{C})$, such that $AB = BA$, there exist $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1 A + \xi_2 B$ is invertible, if and only if $(0, 0) \notin Spec(A, B)$.*

Hence, assuming $(\lambda_1, \lambda_2) \notin Spec(A_1, A_2)$, we may choose ξ_1, ξ_2 such that $\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2)$ is invertible, so that $x_0 = (\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\tilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0$.

Transfer functions of operator vessels

This implies that
$y_0 = (D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\tilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2))u_0$. Hence, the transfer function of a vessel $\mathcal{B}$ is given by

$$S_{\mathcal{B}}(\lambda_1, \lambda_2) = D + C(\xi_1(\lambda_1 I - A_1) + \xi_2(\lambda_2 I - A_2))^{-1}\tilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)$$

Note that for a u_0 with u admissible input signal, this is independent of ξ_1, ξ_2.

Controllability, observability and minimal vessels

Let $\mathcal{C}$ denote the controllable subspace, i.e the space of all vectors $h \in \mathcal{H}$ such that there exist an admissible input u for which the state function x satisfies, $x(0,0) = 0$, and $x(t_1, t_2) = h$ for some $(t_1, t_2) \in \mathbb{R}^2$. A vessel $\mathcal{B}$ is called controllable if $\mathcal{C} = \mathcal{H}$.
Similarly, let $\mathcal{O}^\perp$ denote the unobservable subspace, i.e the subspace of all vectors $h \in \mathcal{H}$ such that the unique solution (u, x, y) of the system of equations associated to the vessel, with $x(0,0) = h$, and $u \equiv 0$ has $y \equiv 0$. A vessel $\mathcal{B}$ is called observable if $\mathcal{O}^\perp = 0$.
A vessel is called minimal if it is both controllable and observable. In the sequel, all vessels are assumed to be minimal.

The discriminant curve of a vessel

Assume now that $\dim \mathcal{E} = \dim \tilde{\mathcal{E}}$, and $\dim \mathcal{E}_* = \dim \tilde{\mathcal{E}}_*$. We have seen that a frequency function u satisfies the compatibility PDE if and only if $(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_0 = 0$. We define $E_{in}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$ and $E_{out}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$ and also $p_{in}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$, $p_{out}(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*)$.

Theorem 3.1. *(Livsic-Kravitsky): $p_{in} \equiv \lambda p_{out}$ for some $\lambda \in \mathbb{C}^\times$.*

Let $p = p_{in} = p_{out}$.

Theorem 3.2. *(Livsic-Kravitsky): The generalized Cayley-Hamilton theorem for vessels: $p(A_1, A_2) = 0$.*

Let $C_0 = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : p(\lambda_1, \lambda_2) = 0\}$. The curve C_0 (and the associated projective plane curve C) is called the discriminant curve of the vessel $\mathcal{B}$.

Corollary 3.3. *$Spec(A_1, A_2) \subseteq C$.*

The discriminant curve of a vessel

The input and output bundles

In all that follows we make the following assumptions on the curve C: C is smooth, reduced, and irreducible of degree m, and intersects the line at infinity at m distinct points.

Theorem 3.4. *For each $\lambda = (\lambda_1, \lambda_2) \in C$,*
$\dim E_{in}(\lambda_1, \lambda_2) = \dim E_{out}(\lambda_1, \lambda_2) = 1.$

It follows that E_{in} and E_{out} are actually line bundles over the curve C. Furthermore, our construction of the transfer function, shows that $S_{\mathcal{B}}$ is actually a bundle map. In summary, we have:

Theorem 3.5. *For a vessel $\mathcal{B}$, the transfer function $S_{\mathcal{B}} : E_{in} \to E_{out}$ is a meromorphic bundle map, with **poles**$(S) =$ **Spec**(A_1, A_2).*

State feedback

We now formulate state feedback for vessels. Let $\mathcal{B}$ be a vessel, and suppose that $F : \mathcal{H} \to \mathcal{E}$. We may form a new collection

$$\mathcal{B}_F^{\textbf{Closed loop}} = (A_1 + \tilde{B}\sigma_1 F, A_2 + \tilde{B}\sigma_2 F, \tilde{B}, C + DF, D, \tilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

A small calculation shows that

Proposition 4.1. *The collection $\mathcal{B}_F^{\textit{Closed loop}}$ is an operator vessel if and only if F satisfies the following 2 equations:*

$$\begin{gathered} \sigma_2 F A_1 - \sigma_1 F A_2 + \gamma F = 0 \\ \sigma_1 F \tilde{B} \sigma_2 - \sigma_2 F \tilde{B} \sigma_1 = 0 \end{gathered}$$

Such an F is called an admissible feedback for $\mathcal{B}$.

The pole placement problem

We may now formulate the pole placement problem for operator vessels:

Problem 4.2. *Given a vessel $\mathcal{B}$, find an admissible feedback F such that the joint spectrum $Spec(A_1 + \tilde{B}\sigma_1 F, A_2 + \tilde{B}\sigma_2 F)$ (which is equal to $Poles(S_{\mathcal{B}_F^{\text{Closed loop}}})$) is a prescribed effective divisor on the discriminant curve C.*

Recall that an effective divisor on a curve C is a finite formal sum $D = \sum_{p \in C} n_p \cdot p$ with $n_p \in \mathbb{N}$.

The factorization

As might be expected, it turns out that the effect of state feedback takes place in the input space. In fact, for every controllable vessel $\mathcal{B}$ and any admissible feedback F, the transfer function of the closed loop vessel $\mathcal{B}_F^{\textbf{Closed loop}}$ factors

$$E_{\text{in}} \xrightarrow{S^{-1}} E_{\text{in}} \xrightarrow{T} E_{\text{out}}$$

where T is the transfer function of the open loop vessel, and S is the transfer function of a vessel $\mathcal{B}_F^{\textbf{Controller}}$ whose construction will be explained in the sequel.

The Ball-Vinnikov realization theorem

Suppose $\sigma_1, \sigma_2, \gamma, \gamma_* \in M_n(\mathbb{C})$ are given matrices, and suppose that $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ is a polynomial defining a reduced smooth irreducible curve C which intersects the line at infinity at $(m = \deg C)$ distinct points. Let $E_{\text{in}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$ and $E_{\text{out}}(\lambda_1, \lambda_2) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma_*)$ be line bundles over C.

Theorem 5.1. *(Ball-Vinnikov): Given any meromorphic bundle map $S : E_{in} \to E_{out}$ such that S acts as the identity operator at all points of C at infinity. Then there exist a unique (up to state space isomorphism) minimal vessel $\mathcal{B}$ with the same determinantal representations and with $D = \tilde{D} = I$ such that $S_{\mathcal{B}} = S$.*

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

In order to solve the pole placement problem, we must analyze 2 different questions:

1. Determine what are the admissible feedbacks (if any)
2. Learn to control the joint spectrum $\operatorname{Spec}(A_1 + \tilde{B}\sigma_1 F, A_2 + \tilde{B}\sigma_2 F)$.

The answer to both of these problems is given by the following object: Given a vessel $\mathcal{B}$ and any operator (not necessarily admissible) $F : \mathcal{H} \to \mathcal{E}$, define a collection

$$\mathcal{B}_F^{\textbf{Controller}} = (A_1, A_2, \tilde{B}, -F, I, I, \sigma_1, \sigma_2, \gamma, \sigma_1, \sigma_2, \gamma)$$

The vessel $\mathcal{B}_F^{\textbf{Controller}}$

Theorem 5.2. *An operator F is an admissible feedback for $\mathcal{B}$ if and only if $\mathcal{B}_F^{\textbf{Controller}}$ is a vessel. Moreover, if F is admissible and S is the transfer function of $\mathcal{B}_F^{\textbf{Controller}}$ then $Poles(S) = Poles(\mathcal{B})$, and $Zeros(S) = Poles(\mathcal{B}_F^{\textbf{Closed loop}})$.*

Proof.

The first assertion is an easy verification. It is also immediate that $\text{Poles}(S) = \text{Poles}(\mathcal{B})$. For the last claim, define the collection

$$\mathcal{V} = (A_1 + \tilde{B}\sigma_1 F, A_2 + \tilde{B}\sigma_2 F, \tilde{B}, -F, I, I, \sigma_1, \sigma_2, \gamma, \sigma_1, \sigma_2, \gamma)$$

One may verify that this is a vessel. It is clear that $\text{Poles}(\mathcal{V}) = \text{Poles}(\mathcal{B}_F^{\textbf{Closed loop}})$. To finish the proof, one checks that $S_{\mathcal{V}} \cdot S = I$. In other words, $\mathcal{V}$ is the inverse of the vessel $\mathcal{B}_F^{\textbf{Controller}}$, so that $\text{Zeros}(S) = \text{Poles}(\mathcal{V})$. In general, for a vessel $\mathcal{B}$ with D invertible, there always exist an inverse vessel $\mathcal{B}^{-1}$ with $S_{\mathcal{B}^{-1}} = (S_{\mathcal{B}})^{-1}$. □

The main theorem

Theorem 5.3. *Given an effective divisor $D \in Div(C)$, there exist an admissible feedback F such that $poles(\mathcal{B}_F^{\textit{Closed loop}}) = D$ if and only if there exist a rational function $f \in K(C)$ such that for every $x \in C \cap L_\infty$, $f(x) = 1$, such that $Poles(f) = Poles(\mathcal{B})$ and $Zeros(f) = D$..*

If such an F exists, $\mathcal{B}_F^{\textbf{Controller}}$ is a vessel with transfer function S. But $S : E_{\text{in}} \to E_{\text{in}}$, has all the required properties, and giving a function from a line bundle to itself is the same as giving a rational function on the curve. For the converse, suppose such an f is given. We may define a function $S : E_{in} \to E_{in}$ by $S(p, v) = f(p) \cdot v$. Such an S satisfies all the conditions of the Ball-Vinnikov realization theorem, so that there exist a vessel $\mathcal{V}$ with transfer function equal to S.

The main theorem

It is now possible to show (using state-space isomorphism, minimality, etc...) that $\mathcal{V}$ is of the form $\mathcal{V} = \mathcal{B}^{\textbf{Controller}}_{-C(\mathcal{V})}$, so that the operator $F = -C(\mathcal{V})$ is the required feedback. $\square$

Genus 0 case

We now show how to obtain information about pole placement in specific examples.

Example 6.1. Suppose $\mathcal{B}$ is a vessel, such that the discriminant curve C is of genus 0. Let $m = \deg C$. One has $m = 1$ or $m = 2$. Suppose $m = 1$. Then C is actually a line. Hence, C intersects the line at infinity at precisely one point. In this case, the interpolation problem of finding a rational function which is 1 at infinity, having prescribed zeros and poles may always be solved, so we can place poles one the curve arbitrarily. This is not surprising, as vessels with $m = 0$ are the same as classical linear systems, so our theorem is indeed a generalization of the classical pole placement.

Genus 0 case

Example 6.2. Suppose now $g = 0$ and $m = 2$. In this case, C is a conic, and it intersects the line at infinity at 2 distinct points. In this case, one can always place exactly $n - 1$ poles, the last pole is then determined from the other $n - 1$.

General conclusions

Recall that $L(D) = \{f \in K(C) : (f) + D \geq 0\}$, and that $l(D) = \dim L(D)$. Let $P = \mathrm{Spec}(A_1, A_2)$.

Corollary 6.3. *Let $l = l(P - D_\infty)$. If $l = 0$ then no poles could be placed. If $l \geq 1$, then generically, one may place exactly l poles. Moreover, generically, once the l poles were chosen, the rest $n - l$ poles of the system are determined uniquely.*

General conclusions

Corollary 6.4. *Let* $m = \deg C$*,* $n = \dim \mathcal{H}$*. If* $n < m$*, no poles could be placed. If* $n - m > 2g - 2$*, then* $l = n - m + 1 - g$*, so that* $n - m + 1 - g$ *poles could be placed generically.*

Elliptic curves

Example 6.5. As a final example, suppose C is an elliptic curve. In this case, $g = 1$, and $m = 3$. For elliptic curves, the Riemann-Roch theorem implies that $l(D) = \deg(D)$ for all D with $\deg(D) \geq 1$. Hence, if $n \geq 4$, one can always place poles, and, generically, one can place $n - 3$ poles. If $n \leq 2$, no place could be placed. The case where $n = 3$ is a special case. In this case, the possibility of placing a single pole depends on the points of p. The theory of special divisors on elliptic curves then shows:

Theorem 6.6. *If $g = 1$ and $n = 3$, then $l > 0$ (so that one can place poles) if and only if for $p = p_1 + p_2 + p_3$, the points p_1, p_2 and p_3 lie on one line. (In terms of the group of points of the elliptic curve, this just means that $p_1 + p_2 + p_3 = 0$).*

Stability

Let $\mathcal{B}$ be a vessel, and suppose that the compact Riemann surface associated to its discriminant curve X, is a real Riemann surface of dividing type. Given the above theorem about pole placement, the question of stability may be reformulated as follows:

Problem 6.7. *Let P be an effective divisor on X. Does there exist a meromorphic function $f : X \to \mathbb{C}$ such that $f(x) = 1$ for all $x \in X \cap L_\infty$, $(f)_\infty = P$, and $(f)_0 \subseteq X_+$?*